Economia Aziendale Online

Business and Management Sciences
International Quarterly Review

Sustainability of collusion and market transparency in a sequential search market: a generalization

Jacopo De Tullio, Giuseppe Puleio

Pavia, March 31, 2023
Volume 14 - N. 1/2023

DOI: 10.13132/2038-5498/14.1.1-14

www.ea2000.it
www.economiaaziendale.it

PaviaUniversityPress

Electronic ISSN 2038-5498
Reg. Trib. Pavia n. 685/2007 R.S.P.

Economia Aziendale Online – Business and Management Sciences International Quarterly Review
Vol. 14-1/2023 - DOI: 10.13132/2038-5498/14.1.1-14

Sustainability of collusion and market transparency in a sequential search market: a generalization

Jacopo De Tullio

Lecturer in Mathematical Analysis, Probability and Mathematical Statistics, Department of Decision Sciences, Bocconi University, Milan, Italy

Giuseppe Puleio

Academic Fellow, Department of Finance, Bocconi University, Milan, Italy

Corresponding Author:

Jacopo De Tullio
Università Bocconi
Via Roentgen, 1 - 20136 Milano
jacopo.detullio@unibocconi.it

Cite as:

De Tullio, J., & Puleio, G. (2023). Sustainability of collusion and market transparency in a sequential search market: a generalization. *Economia Aziendale Online, 14*(1), 1-14.

Section: *Refereed Paper*

Received: January 2023
Published: 31/03/2023

ABSTRACT

The present work generalizes the analytical results of Petrikaite (2016) to a market where more than two firms interact. In other words, we show that, for a generic number of firms in a homogeneous goods market where a continuum of buyers searches sequentially, the relationship between the minimum discount factor which allows the sellers to collude and the share of buyers with null search cost is non-monotonic, reaching a unique interior point of minimum. The first section discusses the motivation of our work and exposes the related literature. The second section summarizes the model of Petrikaite (2016). The third section presents the analytical computations and the mathematical reasoning needed for our generalization, which mainly relies on the Leibniz rule for differentiation under the integral sign and the Bounded Convergence Theorem. The fourth section offers policy implications of market design and suggestions for further research.

Il presente lavoro generalizza i risultati analitici di Petrikaite (2016) a un mercato in cui interagiscono più di due imprese. In altre parole, mostriamo che, per un generico numero di imprese in un mercato di beni omogeneo, in cui un continuum di acquirenti cerca, in sequenza, la relazione tra il minimo fattore di sconto che consente ai venditori di colludere, e la quota di acquirenti con costo di ricerca nullo, è non monotono, raggiungendo un unico punto interno di minimo. Il primo paragrafo discute la motivazione del nostro lavoro ed espone la relativa letteratura. Il secondo paragrafo riassume il modello di Petrikaite (2016). Il terzo paragrafo presenta i calcoli analitici e il ragionamento matematico necessari per la nostra generalizzazione, che si basa principalmente sulla regola di Leibniz per la differenziazione sotto il segno integrale e il teorema di convergenza limitata. Il quarto paragrafo offre implicazioni di politica del design del mercato e suggerimenti per ulteriori ricerche.

Keywords: Game Theory, Collusion, Industrial Organization, Market Design, Functional Analysis, Bounded Convergence Theorem.

1 – Introduction and related literature

The characterization of oligopolistic equilibria where consumers display search costs or similar informational frictions is a traditional topic of the theory of Industrial Organization[1].

This area of research experienced a renewed interest by the scholar community due to the advent of the internet economics: the search behaviour of potential buyers in digital economic interactions has fundamental strategic implications for both the platforms – like Amazon, eBay, Google Flights – which set up the market and, of course, for the oligopolistic sellers – such as publishing houses, real estate rental agencies or airline companies[2].

A recent strand of the literature aims at investigating the relationship between demand side market transparency – defined as the consumers awareness of the different quotations and varieties offered in the market, which, clearly, results from their search behaviour – and the incentives of the sellers to enforce a collusive agreement. Indeed, the relationship between intensity of search by the buyers and collusive incentives by the sellers is a priori ambiguous, under the lenses of classical game theory. If, on one hand, buyers with higher propensity to search would undermine the stability of a supposedly established cartel – as a firm which deviates from the cartel agreement, setting a slightly lower price, is able to attract a higher fraction of the demand – on the other hand, firms incentives to constitute a cartel might be higher in the first place, given that a more elastic demand would drive equilibrium prices closer to marginal cost if they were not to collude.

In the light of the previous reasoning, one should not be surprised to observe a non-monotonic relationship between demand side search intensity and offer side collusive incentives. According to which of the two arguments dominates, the relationship will be decreasing or increasing, for a given demand transparency to start with. This intuition is confirmed by the result by Petrikaite (2016), which shows that the likelihood of cartel formation can be expressed as a U-shaped function of the proportion of perfectly informed buyers – to be taken as an indicator of search intensity – when there are only two firms in a market with homogeneous goods. Our main contribution is to generalize her result, by showing that the same U-shaped pattern occurs in a generic market where $N>2$ companies interact.

Prominent studies that deal with the impact of market transparency on collusion sustainability are Montag and Winter (2020), Nilsson *et al.* (1999), Schultz (2017) and Cabral *et al.* (2019). Montag and Winter (2020) extend the model of Petrikaite (2016) to include the impact of supply side transparency, which is relevant if also firms, in addition to consumers, cannot perfectly observe the prices chosen by their competitors. They show that the U-shaped relationship between demand side transparency and collusion likelihood is preserved in their setting, for any given level of producer side transparency. Cabral *et al.* (2019), Nilsson *et al.* (1999), and Schultz (2017) assumed, differently from Petrikaite (2016), that buyers do not follow a sequential search rule. Schultz (2017) finds that, under this condition, increasing a common factor of transparency, affecting both producers and consumers simultaneously, is always anti-

---

[1] See Diamond (1971), Burdett and Judd (1983), Varian (1980), Stahl (1989), Dana (1994), Janssen *et al.* (2005).

[2] In this regard, see Armstrong (2015), Anderson and Renault (2018), Levin *et al.* (2018). For excellent surveys, see Levin (2011), Goldfarb and Tucker (2019) and Calvano and Polo (2021).

competitive in homogeneous goods markets, strenghtening collusion incentives. Cabral *et al.* (2019) introduce the role of the antitrust authority (AA) into the picture: they show that a U-shaped relationship exists between the average price of the economy and the share of perfectly informed consumers; intuitively, for extremely low values of this parameter firms will find competition more profitable, while, as demand-side transparency increases, the collusive allocation will guarantee higher profits to the undertakings, notwithstanding the possibility of penalties by the AA. Nilsson (1999) shows that an increase in search cost bore by imperfectly informed buyers will reduce the likelihood of collusion, while an increase in the share of perfectly informed buyers either does not affect collusion sustainability or affects it negatively, i.e. it is weakly procompetitive. However, in his setting, captive consumers do not search sequentially; they, instead, have the possibility of becoming aware of all the quotations once having paid a fixed search cost.

The rest of the work is divided as follows: section 2 recalls the model of Petrikaite (2016), section 3 presents our generalization, section 4 offers concluding remarks. In APPENDIX 1 (Section 5), we present an alternative proof for one of the main analytical results of the paper.

2 – The model

The work of Petrikaite (2016) analyses the sustainability of the monopoly collusive allocation as a *Subgame Perfect Equilibrium* (SPE) of the super-game extension[3] of the model of costly sequential search described by Janssen *et al.* (2005)[4], in the case of a duopoly.

A well-established result of the theory of Industrial Organization asserts that in an infinitely repeated oligopoly game with almost perfect monitoring – i.e. where prices of period $t-1$ are public knowledge at the start of period t – the monopoly collusive allocation constitutes a SPE provided that the inter-temporal discount factor $\delta \in (0, 1)$ is not inferior than a certain critical value. Indeed, if firms are sufficiently patient, they are willing to renounce to the short-term gains that they could achieve best-responding against the collusive agreement, in order to preserve the future *supracompetitive* margins that collusion is expected to bring about in the long run[5]. The minimum discount factor δ^* is related to the one-shot Nash Equilibrium profits π^*, the collusive profits π^c and the deviation profits π^d by the following equation:

$$\delta^* = \frac{\pi^d - \pi^c}{\pi^d - \pi^*} \in (0, 1)$$

[3] By super-game extension, we indicate the repetition of an identical static game for a countably infinite number of periods. See, *inter alia*, Fudenberg and Maskin (1986).

[4] We refer to the case of full consumer participation addressed by Janssen *et al.* (2005). Within this framework, this model should be thought of as the inelastic demand version of the seminal model of Stahl (1989).

[5] This is the classic model of tacit collusion when firms play grim-trigger strategies, i.e. after a deviation from the collusive agreement, cartel members revert to play Nash Equilibrium actions for each of the following periods. See, *inter alia*, Motta (2004) and Tirole (1988).

The one-shot game of Janssen *et al.* (2005) works as follows: consumers search sequentially in a homogeneous market with perfect recall; a portion λ of them can become aware of all the different quotations listed by the firms at no cost, while the remaining part of buyers suffer a constant search cost s, smaller than their maximum willingness to pay v, for each price they might desire to observe, except for the first one. Demand is inelastic: each consumer purchases one unit of output, as long as the minimum price of which it is aware does not exceed its maximum willingness to pay. Firms set price simultaneously in order to maximize profits, committing to the pricing strategy while buyers seek through the market, but anticipating their optimal search behavior. Janssen *et al.* (2005) focus on the unique symmetric equilibrium in mixed actions, where each firm charges the same distribution of prices[6].

Given this environment, in a monopoly collusive allocation it holds that $\pi^c = \frac{v}{N}$, $\pi^d = v\left(\frac{1-\lambda}{N} + \lambda\right)$ and $\pi^* = \frac{p^*(1-\lambda)}{N}$ where N represents the number of firms in the market and

$$p^* = \frac{s}{1 - \int_0^1 \frac{dy}{1+\frac{\lambda}{1-\lambda}Ny^{N-1}}} = \frac{s}{1 - G(\lambda; N)}$$

represents the endogenous reservation price, which is defined as the price that makes imperfectly informed buyers indifferent between purchasing at the price at hand or carrying out a further search through the market.

Indeed, while suffering a cost to observe each individual price in the market, imperfectly informed buyers hold nevertheless a correct expectation of the distribution of prices, i.e. the equilibrium distribution of prices. N, λ, v and s are all exogenous, and it is assumed that $0 < s < v$ and $1 > \lambda \geq \hat{\lambda}$, so that a reservation price equilibrium is achieved[7], while $y \in [0, 1]$ is an auxiliary variable derived from the equilibrium distribution of prices.

In the light of what has been specified above, it is possible to write:

$$\delta^* = \frac{\lambda(N-1)}{1 + \lambda(N-1) - \frac{p^*(1-\lambda)}{v}} \qquad [1]$$

and

$$\frac{\partial \delta^*}{\partial \lambda} = \frac{1}{(\pi^d - \pi^*)^2}\left[\frac{\partial \pi^*}{\partial \lambda}(\pi^d - \pi^c) + \frac{\partial \pi^d}{\partial \lambda}(\pi^c - \pi^*)\right] =$$
$$= \frac{1}{(\pi^d - \pi^*)^2}\left[\frac{v^2}{N^2}(N-1)\left(v - p^* + \frac{\partial p^*}{\partial \lambda}(1-\lambda)\lambda\right)\right]$$

[6] It can be shown, indeed, that a symmetric equilibrium in pure actions, where each firm sets the same price, does not exist in this setting. See Sthal (1989) and Janssen *et al.* (2005).

[7] A reservation price equilibrium exists if and only if $\lambda \geq \hat{\lambda}$ where $\hat{\lambda}$ satisfies
$1 - \int_0^1 \frac{dy}{1+\frac{\hat{\lambda}}{1-\hat{\lambda}}Ny^{N-1}} = \frac{s}{v}$, given $s \in (0, v)$.

The reservation price will correspond to the upper bound of the prices' distribution; see Pennerstorfer *et al.* (2020) and Janssen *et al.* (2005) for more details. The case of a no-reservation price equilibrium is significantly less interesting, since in that case the critical discount factor will depend only on the number of firms in the market, being insensitive to variations in the relative number of perfectly informed buyers.

Given $s \in (0, v)$, let

$$\Gamma(\lambda; N) \coloneqq v - p^* + \frac{\partial p^*}{\partial \lambda}(1-\lambda)\lambda$$

Petrikaite (2016) shows that $\frac{\partial \Gamma(\lambda;2)}{\partial \lambda}|_{\lambda \in (0,1)} > 0$; *a fortiori*, $\frac{\partial \Gamma(\lambda;2)}{\partial \lambda}|_{\lambda \in (\hat{\lambda},1)} > 0$. As $\lim_{\lambda \to \hat{\lambda}} \Gamma(\lambda; 2) = v^2\left(\frac{1}{v} - \frac{2\hat{\lambda}}{(1+\hat{\lambda})s}\right) < 0$ and $\lim_{\lambda \to 1} \Gamma(\lambda; 2) = v - s > 0$, it is possible to deduce that, given $N = 2$ and $s \in (0, v)$, $\exists\, \tilde{\lambda}$ such that $\forall\, \lambda \in (\hat{\lambda}, \tilde{\lambda}]$ $\frac{\partial \delta^*}{\partial \lambda} \leq 0$ and $\forall\, \lambda \in (\tilde{\lambda}, 1)$ $\frac{\partial \delta^*}{\partial \lambda} > 0$.

3 – The general case

The aim of this section is to show that the results which apply to a duopoly can be extended to a case where a higher number of firms interact. Provided that the share of shoppers is sufficiently high so that a reservation price equilibrium is achieved, the critical discount factor will be first decreasing and later increasing in this parameter, reaching a unique interior point of minimum. Hence, in what follows, $N > 2$ ($N \in \mathbb{N}$)[8] (See Figure 1 and Figure 2. APPENDIX 2. Section 6).

3.1 – *Deriving the inequality*

Consider again

$$\Gamma(\lambda; N) = v - p^* + \frac{\partial p^*}{\partial \lambda}(1-\lambda)\lambda = \qquad [2]$$

$$= v - p^* - (1-\lambda)\lambda\left(\frac{p^*}{1 - G(\lambda; N)} \cdot \int_0^1 \frac{N y^{N-1}}{\left((1-\lambda) + \lambda N y^{N-1}\right)^2} dy\right)$$

Notice that

$$\int_0^1 \frac{N y^{N-1}}{\left((1-\lambda) + \lambda N y^{N-1}\right)^2} dy = \frac{1}{(1-\lambda)^2} \int_0^1 \frac{N y^{N-2} y}{\left(1 + \frac{\lambda}{1-\lambda} N y^{N-1}\right)^2} dy =$$

$$= \frac{1}{\lambda(N-1)}\left[\frac{G(\lambda; N)\big(1 + \lambda(N-1)\big) - (1-\lambda)}{\big(1 + \lambda(N-1)\big)(1-\lambda)}\right] > 0$$

So, equation [2] can be rewritten as

$$\Gamma(\lambda; N) = v - p^*\left[1 + \left(\frac{G(\lambda; N)\big(1 + \lambda(N-1)\big) - (1-\lambda)}{(N-1)\big(1 + \lambda(N-1)\big)\big(1 - G(\lambda; N)\big)}\right)\right] =$$

$$= v - p^*[1 + H(\lambda; N)]$$

Given that $\lim_{\lambda \to \hat{\lambda}} \Gamma(\lambda; N) = v - v\big[1 + H(\hat{\lambda}; N)\big] < 0$ and $\lim_{\lambda \to 1} \Gamma(\lambda; N) = v - s > 0$, in order to show that equation [1] has a unique internal point of minimum it is sufficient to show that

[8] Clearly, from an economic point of view, it is reasonable to restrict the analysis to $N \leq 10$.

$$\frac{\partial\Gamma(\lambda;N)}{\partial\lambda} = -\left\{\frac{\partial p^*}{\partial\lambda}[1 - H(\lambda;N)] + p^*\frac{\partial H(\lambda;N)}{\partial\lambda}\right\} > 0 \quad \forall\, \lambda \in (\hat{\lambda}, 1) \qquad [3]$$

As $\frac{\partial p^*}{\partial\lambda} < 0\ \ \forall\, \lambda \in (\hat{\lambda}, 1)$ and the reservation price is clearly positive, to show that equation [3] holds it is sufficient to show that $\frac{\partial H(\lambda;N)}{\partial\lambda} < 0\ \ \forall\, \lambda \in (0,1)$.

Algebraic manipulations reveal that

$$\frac{(N-1)^2\big(1+\lambda(N-1)\big)^2\big(1-G(\lambda;N)\big)^2}{N^2}\cdot\frac{\partial H(\lambda;N)}{\partial\lambda} = \left(1 - \frac{G(\lambda;N)}{1-\lambda}\right)$$

so that the condition $G(\lambda;N) \geq 1-\lambda\ \ \forall\, \lambda \in (0,1)$ is sufficient to show that equation [3] holds (See Figure 3. APPENDIX 2. Section 6).

3.2 – *Solving the inequality*

Our ultimate objective is, therefore, to show that:

$$G(\lambda;N) - (1-\lambda) \geq 0\ \ \forall\, \lambda \in (0,1), \forall\, N > 2 \text{ with } N \in \mathbb{N} \qquad [4]$$

To this end, it is necessary to exploit the Leibniz rule[9] and the *Bounded Convergence Theorem* for the Riemann integral[10]. These two results allow us to state that $\lim_{\lambda\to 0} G'(\lambda;N) = -1$, $\lim_{\lambda\to 0} G''(\lambda;N) > 0$ and $\frac{\partial G''(\lambda;N)}{\partial\lambda} < 0\ \ \forall\, \lambda \in (0,1)$. Importantly, it has been already established that $\lim_{\lambda\to 0} G(\lambda;N) = 1$ and $\lim_{\lambda\to 1} G(\lambda;N) = 0$[11].

Let us show first that $\lim_{\lambda\to 0} G'(\lambda;N) = -1$, where, by the Leibniz rule,

$$G'(\lambda;N) = -\int_0^1 \frac{Ny^{N-1}}{[1+\lambda(Ny^{N-1}-1)]^2}\,dy$$

It is useful to write

$$\lim_{\lambda\to 0}\int_0^1 \frac{Ny^{N-1}}{[1+\lambda(Ny^{N-1}-1)]^2}\,dy = \lim_{n\to\infty}\int_0^1 \frac{Ny^{N-1}}{[1+\lambda_n(Ny^{N-1}-1)]^2}\,dy$$

where $\{\lambda_n\}$ is a generic sequence which satisfies $\forall\, n \in \mathbb{N}, \lambda_n \in (0,1)$ and $\lim_{n\to\infty}\lambda_n = 0$.

Clearly, $\forall\, n$

$$f_n(y) = \frac{Ny^{N-1}}{[1+\lambda_n(Ny^{N-1}-1)]^2}$$

is a Riemann-integrable function in $[0,1]$. Moreover, $\{f_n(y)\}$ converges pointwise to $f(y) = Ny^{N-1}$ in $[0,1]$, which is a Riemann-integrable function. Hence, to apply the *Bounded Convergence*

[9] See Mukhopadhyay (2000).

[10] See Gordon (2000).

[11] See Janssen *et al.* (2005) in this regard.

Theorem, it remains to show that the sequence $\{f_n(y)\}$ is uniformly bounded. This motivates the following proposition.

PROPOSITION 1*. The sequence $\{f_n(y)\}$ is uniformly bounded.*

Proof. The proof works by contradiction. Suppose $\exists\, n^*$ such that $\nexists\, M \in \mathbb{R}_+$ which satisfies $|f_{n^*}(y)| = f_{n^*}(y) \leq M$, $\forall\, y \in [0,1]$. Consider, for an arbitrary chosen $\varepsilon \in (0,1)$, the set $A_\varepsilon = \{\lambda_n \notin B_\varepsilon(0)\}$, for any ε, the set A_ε is either empty or finite. Suppose it is empty, so that $\lambda_{n^*} \in B_\varepsilon(0)$. Then, $\forall\, y \in \left[0, \frac{1}{N}^{\frac{1}{N-1}}\right]$, $\forall\, n$

$$f_\varepsilon(y) = \frac{Ny^{N-1}}{[1+\varepsilon(Ny^{N-1}-1)]^2} \geq f_n(y)$$

While $\forall\, y \in \left[\frac{1}{N}^{\frac{1}{N-1}}, 1\right]$ we have $f(y) = Ny^{N-1} \geq f_n(y)\ \forall\, n$.

As a result, set $M = \max\left\{\max_{y\in\left[0,\frac{1}{N}^{\frac{1}{N-1}}\right]} f_\varepsilon(y), N\right\} \geq f_{n^*}(y)$ and the desired contradiction is reached.

Let A_ε be finite. Then $\exists\, \lambda_m = \max \lambda_n \in A_\varepsilon$. Then, define

$$K = \max_{y\in\left[0,\frac{1}{N}^{\frac{1}{N-1}}\right]} f_{\lambda_m}(y)$$

Setting $M = \max\{K, N\}$ we reached the desired contradiction, independently from whether $\lambda_{n^*} \in A_\varepsilon$ or not. The existence of the upper bound relies on the continuity of the functions $f_\varepsilon(y)$ and $f_{\lambda_m}(y)$, which are defined on a compact interval. ■

Therefore, from the *Bounded Convergence Theorem*:

$$\lim_{n\to\infty}\int_0^1 \frac{Ny^{N-1}}{[1+\lambda_n(Ny^{N-1}-1)]^2}dy = \int_0^1 \lim_{n\to\infty}\frac{Ny^{N-1}}{[1+\lambda_n(Ny^{N-1}-1)]^2}dy = \int_0^1 Ny^{N-1}dy = 1 \qquad [5]$$

and $\lim_{\lambda\to 0} G'(\lambda; N) = -1$.

Let us show the results for $G''(\lambda; N)$. Given N and y, $f_{N,y}(\lambda) = \frac{Ny^{N-1}}{[1+\lambda(Ny^{N-1}-1)]^2}$ is differentiable for λ in $(0,1)$. Thanks to this observation, it is licit to apply the Leibniz Rule, concluding that:

$$G''(\lambda; N) = \int_0^1 \frac{2Ny^{N-1}(Ny^{N-1}-1)}{[1+\lambda(Ny^{N-1}-1)]^3}dy$$

To evaluate $\lim_{\lambda\to 0} G''(\lambda; N)$, define the sequence, for a given N, $\{g_n\}$ with generic element

$$g_n(y) = \frac{2Ny^{N-1}(Ny^{N-1}-1)}{[1+\lambda(Ny^{N-1}-1)]^3}$$

where $\{\lambda_n\}$ is a generic sequence which satisfies $\forall\, n \in \mathbb{N}$, $\lambda_n \in (0,1)$ and $\lim_{n\to\infty} \lambda_n = 0$. We have that $\{g_n\}$ is a sequence of Riemann integrable functions, since every g_n is a continuous function defined on a compact interval. Moreover, the sequence converges pointwise to $g(y) = 2Ny^{N-1}(Ny^{N-1} - 1)$ in $[0,1]$, which is a Riemann Integrable function. Uniform boundedness[12] of $\{g_n\}$ ensures that the *Bounded Convergence Theorem* holds, so that we are allowed to write:

$$\lim_{n\to\infty} \int_0^1 \frac{2Ny^{N-1}(Ny^{N-1}-1)}{[1+\lambda_n(Ny^{N-1}-1)]^3}dy = \int_0^1 2Ny^{N-1}(Ny^{N-1}-1)dy = \qquad [6]$$

$$= 2N\left(\frac{N}{2(N-1)+1} - \frac{1}{N}\right) > 0 \quad \forall\, N > 2$$

so that $\lim_{\lambda\to 0} G''(\lambda; N) > 0$. Finally, it is possible to notice that $G''(\lambda; N)$ is strictly decreasing in $\lambda \in (0,1)$. For every given N and every given y, $f_{N,y}(\lambda) = \frac{2Ny^{N-1}(Ny^{N-1}-1)}{[1+\lambda(Ny^{N-1}-1)]^3}$ is differentiable in $\lambda \in (0,1)$. By the Leibniz Rule,

$$\frac{\partial}{\partial\lambda}\int_0^1 \frac{2Ny^{N-1}(Ny^{N-1}-1)}{[1+\lambda(Ny^{N-1}-1)]^3}dy = \int_0^1 -\frac{6Ny^{N-1}(Ny^{N-1}-1)^2}{[1+\lambda(Ny^{N-1}-1)]^4}dy$$

Given that

$$\int_0^1 \frac{Ny^{N-1}(Ny^{N-1}-1)^2}{[1+\lambda(Ny^{N-1}-1)]^4}dy > \int_0^1 \frac{Ny^{N-1}(Ny^{N-1}-1)^2}{[1+\lambda(N-1)]^4}dy =$$

$$= \frac{1}{[1+\lambda(N-1)]^4}\left\{\frac{N^3}{3(N-1)+1} + 1 - \frac{2N^2}{2(N-1)+1}\right\} > 0 \quad \forall\, N > 2$$

then $\frac{\partial}{\partial\lambda} G''(\lambda; N) < 0 \;\; \forall\lambda \in (0,1)$. This result is crucial in motivating the following proposition.

PROPOSITION 2. $\forall\, N > 2$ *there exists a unique* $\lambda_N^f \in (0,1)$ *such that* $G''(\lambda_N^f; N) = 0$.

Proof. Uniqueness stems directly from the strict and decreasing monotonicity of the function $G''(\lambda; N)$. Suppose that the inflection point did not exist. Then, $G''(\lambda; N) > 0 \;\; \forall\lambda \in (0,1)$ and, as a consequence, $G'(\lambda; N) > -1 \;\; \forall\lambda \in (0,1)$.

Given that $\lim_{\lambda\to 0} G(\lambda; N) = 1$, this would contradict the result of $\lim_{\lambda\to 1} G(\lambda; N) = 0$. ■

So far, we have showed that $\forall\, N \in \mathbb{N}$, exists a unique $\lambda_N^f \in (0,1)$ such that $G'(\lambda; N) > -1 \;\; \forall\lambda \in (0, \lambda_N^f]$ and the function $G(\lambda; N)$ is strictly concave in $(\lambda_N^f, 1)$.

These preliminary conditions allow us to show the desired inequality.

PROPOSITION 3.

$$G(\lambda; N) - (1-\lambda) \geq 0 \quad \forall\, \lambda \in (0,1), \forall\, N > 2$$

[12] The proof is identical in spirit to the one of ***PROPOSITION 1***.

Proof. Suppose $\exists\, \bar{\lambda} \in \left(0, \lambda_N^f\right]$ such that $G\left(\bar{\lambda}; N\right) < 1 - \bar{\lambda}$; this would contradict the results of $\lim_{\lambda \to 0} G(\lambda; N) = 1$ and $G'(\lambda; N) > -1\ \ \forall \lambda \in \left(0, \lambda_N^f\right]$. Suppose $\exists \hat{\lambda} \in \left(\lambda_N^f, 1\right)$ such that $G\left(\hat{\lambda}; N\right) < 1 - \hat{\lambda}$. Then, given that $G\left(\lambda_N^f; N\right) > \left(1 - \lambda_N^f\right)$ and $G''^{(\lambda;N)} < 0\ \ \forall \lambda \in \left(\lambda_N^f, 1\right)$, the condition $\lim_{\lambda \to 1} G(\lambda; N) = 0$ would be impossible to be satisfied. ∎

Thank to ***Proposition 3***, we are allowed to state that, given $N \in \mathbb{N}$ and $s \in (0, v)$, $\exists\, \tilde{\lambda}$ such that $\forall \lambda \in \left(\hat{\lambda}, \tilde{\lambda}\right]$ we have $\frac{\partial \delta^*}{\partial \lambda} \leq 0$ and $\forall \lambda \in \left(\tilde{\lambda}, 1\right)$ we have $\frac{\partial \delta^*}{\partial \lambda} > 0$.

4 – Concluding remarks

Our work extends the literature on collusion in homogeneous goods markets with buyers sequential search by showing that the number of firms do not affect, qualitatively, the relationship between buyers search intensity and sellers incentives to collude.

As a result, competition policy implications which arise in the case of a duopoly are confirmed for an arbitrary higher number of firms: enriching the consumers ability to retrieve different offers in the market does not always lower the concerns of an Antitrust Authority willing to minimize the likelihood of collusion. Specifically, if search intensity is low to start with, a marginal increase in demand side transparency should call for higher penalties by the competition authority. For the same argument, policymakers who aim at reducing the opacity of homogeneous markets should favour significant increases in transparency, targeting a Bertrand-type market, in order to ensure that the higher search intensity would not bring about anticompetitive effects.

As far as concerns methodology, our approach offers different applications of the *Convergence Theorems* presented by Gordon (2000), which allow the interchange of the operations of limit and integration. The arguments employed to construct the proofs (see ***Proposition 1*** and the Appendix 1 can be reasonably generalized in similar analysis problems which deal with sequences of continuous functions in compact domains.

The U-shaped relationship between search intensity and collusion sustainability arises from the characterization of the per-period payoffs available by the firm, as a function of the share of perfectly informed buyers in the market. Indeed, while Nash Equilibrium (NE) payoffs decrease exponentially with the share of perfectly informed buyers, deviation payoffs increase linearly with this parameter, so that, as search intensity increases, the marginal impact on NE profits will be negligible, while the marginal impact on deviation profits is constant. As already pointed out by Nilsson (1999) and Schultz (2017), the same relationship does not hold if buyers search is assumed to be non-sequential[13]. Furthermore, our approach and the approach of Petrikaite (2016), which builds upon the framework of super games, does not consider the possibility for

[13] Caution should employed when distinguishing between sequential and non-sequential search. Indeed, while commonly used platforms allow buyers to access a comprehensive list of prices, one could still argue that buyers will *process* such information in a sequential manner, as they scroll through the webpage or the magazine.

the firms to not perfectly anticipate future variations in buyers search intensity. By definition of supergame, each period game is identical, so that variations in demand side transparency are more suited to represent comparisons between collusion sustainability across *different* markets- identical except for the proportion of informed buyers- rather than representing comparisons of *intertemporal* collusion incentives within the same market, when buyers become (exogenously) more or less active searchers. In other words, if firms do not expect buyers to search more intensely in the future, collusion might be chosen in the current period, as firms do not foresee that implementing a cartel later on, with a higher demand side transparency, would require such a high level of patience to be unsustainable. This reasoning motivates the interest for analyzing collusion sustainability in a Bayesian game with a stochastic pattern of demand side transparency, to complement the analysis presented in this paper under a super game setting.

Furthermore, our analysis assumed that collusion is enforced through grim-trigger strategies, which specify that, after a deviation by one of the cartel members, collusion is never restarted and Nash Equilibrium actions are played in any of the following periods.

Nevertheless, in a more realistic setting of stochastic demand, a firm who observes unexpectedly low sales cannot infer with certainty whether some of the cartel members did not respect the agreement[14]. Considering *stick-and-carrot* strategies *a la* Abreu (1986,1988) – where deviators from the collusive agreement are punished for a limited number of periods, after which collusion is restarted – would attribute to further research on the topic a more grounded behavioral and empirical perspective.

5 – APPENDIX 1 – Alternative proof for equation [5] and equation [6]

Instead of employing pointwise convergence, Riemann integrability and uniform boundedness of $\{f_n(y)\}$ and $\{g_n(y)\}$, we are allowed to employ uniform convergence to justify the interchange between the operations of limit and integration. Indeed, *Theorem 1* in Gordon (2000) states that, given a sequence $\{f_n\}$ of Riemann-integrable functions defined on $[a,b]$, if f_n coverges uniformly to f on $[a,b]$, then f is Riemann-integrable on $[a,b]$ and $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$. We show that *Theorem 1* applies to both

$$f_n(y) = \frac{Ny^{N-1}}{[1+\lambda_n(Ny^{N-1}-1)]^2}$$

and

$$g_n(y) = \frac{2Ny^{N-1}(Ny^{N-1}-1)}{[1+\lambda_n(Ny^{N-1}-1)]^3}$$

where $y \in [0,1]$. Recall that $\{\lambda_n\}$ is a generic sequence which satisfies $\lambda_n \in (0,1)\ \forall n$ and $\lim_{n\to\infty}\lambda_n = 0$. Notice, indeed, $\{f_n\}$ is a sequence of continuous functions defined on $[0,1]$, so that, *a fortiori*, it is a sequence of Riemann-integrable functions. We show that $f_n(y)$ converges uniformly to

[14] See the seminal paper by Green and Porter (1984).

$f(y) = Ny^{N-1}$. By definition, $f_n \to f$ uniformly if and only if the sequence of real numbers $d_n \to 0$ as $n \to +\infty$, where

$$d_n = \sup_{y\in[0,1]} \left| \frac{Ny^{N-1}}{[1+\lambda_n(Ny^{N-1}-1)]^2} - Ny^{N-1} \right| =$$

$$= \sup_{y\in[0,1]} \left| Ny^{N-1}\left[\frac{Ny^{N-1}}{[1+\lambda_n(Ny^{N-1}-1)]^2} - 1\right]\right| = \sup_{y\in[0,1]} \left|\tilde{d}_n(y)\right|$$

Some observations are worth of notice. First, $\{\tilde{d}_n(y)\}$ is a sequence of continuous functions, given that $\forall n$, $\tilde{d}_n$ can be expressed as a product of continuous functions. As a result, $\{|\tilde{d}_n|\}$ is a sequence of continuous functions on [0,1]. Moreover, it is clear that, $\forall n, \tilde{d}_n \geq 0\ \ \forall y \in \left[0, \frac{1}{N}^{\frac{1}{N-1}}\right]$, while $\forall y \in \left[\frac{1}{N}^{\frac{1}{N-1}}, 1\right]$ we have $\tilde{d}_n \leq 0$. As a result:

$$d_n = \max_{y\in[0,1]} \left|\tilde{d}_n(y)\right| = \max\left\{ \max_{y\in\left[0,\frac{1}{N}^{\frac{1}{N-1}}\right]} \tilde{d}_n(y)\,,\ \max_{y\in\left[\frac{1}{N}^{\frac{1}{N-1}},1\right]} -\tilde{d}_n(y)\right\}$$

As a consequence,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \max\{\tilde{d}_n(y_+^*(n)), -\tilde{d}_n(y_-^*(n))\} =$$

$$= \max\left\{\lim_{n\to\infty} \tilde{d}_n(y_+^*(n))\,,\ \lim_{n\to\infty} -\tilde{d}_n(y_-^*(n))\right\}$$

where $y_+^*(n) \in argmax\left(\tilde{d}_n(y)\right)$ and $y_-^*(n) \in argmax\left(-\tilde{d}_n(y)\right)$ on the respective domains. The notation emphasizes that the argmax set might depend on n. By continuity and domain compactness of $\tilde{d}_n$ and of $-\tilde{d}_n$, it holds that, $\forall n$, $\exists M \in \mathbb{R}_+$ such that $M \geq \max\left\{N\left(y_+^*(n)\right)^{N-1}, N\left(y_-^*(n)\right)^{N-1}\right\}$. As a consequence,

$$\lim_{n\to\infty} \tilde{d}_n(y_+^*(n)) = \lim_{n\to\infty} -\tilde{d}_n(y_-^*(n)) = 0$$

and

$$\lim_{n\to\infty} d_n = 0$$

As desired. An analogous argument applies to the sequence g_n. In this case,

$$d_n = \max_{y\in[0,1]} \left| \frac{2Ny^{N-1}(Ny^{N-1}-1)}{[1+\lambda_n(Ny^{N-1}-1)]^3} - 2Ny^{N-1}(Ny^{N-1}-1)\right| =$$

$$= \max_{y\in[0,1]} \left| 2Ny^{N-1}(Ny^{N-1}-1)\left(\frac{1}{[1+\lambda_n(Ny^{N-1}-1)]^3} - 1\right)\right| =$$

$$\max_{y\in[0,1]} \left|\tilde{d}_n(y)\right|$$

Notice that $\tilde{d}_n(y) \leq 0\ \forall y \in [0,1]$, so that we can write

$$\lim_{n\to\infty} d_n(y) = \lim_{n\to\infty} \max_{y\in[0,1]} -\tilde{d}_n(y) = \lim_{n\to\infty} -\tilde{d}_n(y^*(n)) = 0$$

where the last equality follows from the continuity and domain compactness of $-\tilde{d}_n$, so that the sequence $\left\{N\left(y^*(n)\right)^{N-1}\right\}$ is bounded.

6 – APPENDIX 2. List of Figures

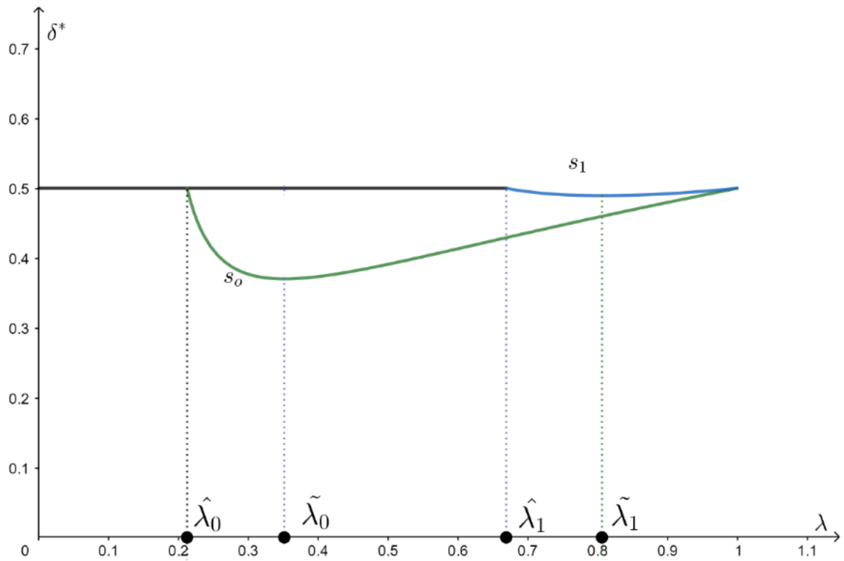

Figure 1 – Critical discount factor as a function of the proportion of shoppers in a duopoly. Here, $v = 1, s_0 = 0.2, s_1 = 0.6$.

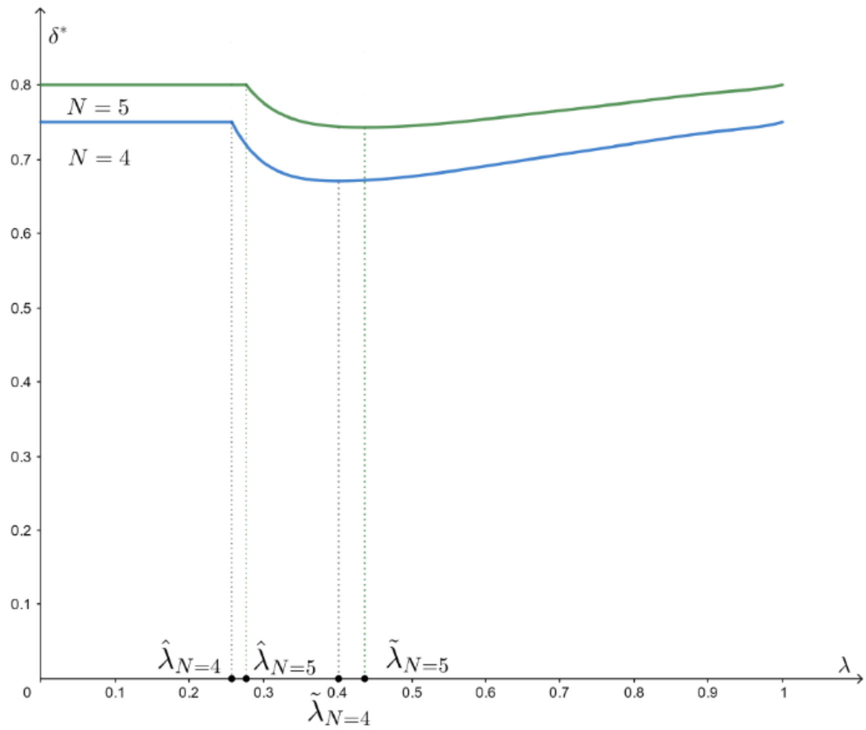

Figure 2 – Critical discount factor as a function of the proportion of shoppers when $N = 4$ and $N = 5$. Here, $v = 1$ and $s = 0.2$ fixed. Numerical approximations show that the duopoly pattern is confirmed also when a higher number of firms interacts.

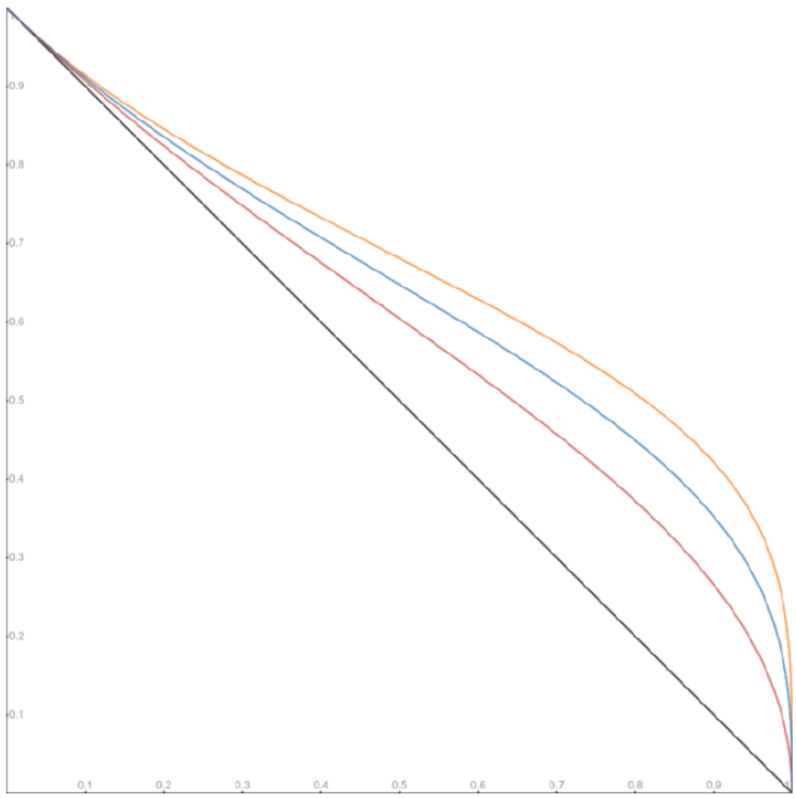

Figure 3 – Numerical approximations confirm the desired inequality. The black line indicates $\boldsymbol{f(\lambda) = 1 - \lambda \leq G(\lambda; 3) \leq G(\lambda; 4) \leq G(\lambda; 5)}$, where $\boldsymbol{\lambda \in (0, 1)}$.

7 – References

Abreu, D., Pearce, D. & Stacchetti, E. (1986). Optimal cartel equilibria with imperfect monitoring. *Journal of Economic Theory*, 39(1), 251–269.

Abreu, D. (1988). On the theory of infinitely repeated games with discounting. *Econometrica: Journal of the Econometric Society*, *56*(2), 383–396.

Anderson, S. & Renault, R. (2018). Firm Pricing with Consumer Search. in Corchón, L. & Marini, M. (eds.), *Handbook of Game Theory and Industrial Organization*, Vol. 2, Edward Elgar, 177–224.

Armstrong, M. (2015). Search and Ripoff Externalities. *Review of Industrial Organization, 47(3)*, 273–302.

Burdett, K. & Judd, K.L. (1983). Equilibrium Price Dispersion. *Econometrica, 51(4)*, 955–969.

Cabral, L., Dominik Schober, D. & Woll, O. (2019). Search and equilibrium prices: Theory and evidence from retail diesel. *ZEW-Centre for European Economic Research Discussion Paper*.

Calvano, E. & Polo, M. (2021). Market power, competition and innovation in digital markets: A survey. *Information Economics and Policy, 54, 100853*.

Dinerstein, M.,Einav, L., Levin, J. & Sundaresan, N. (2018). Consumer Price Search and Platform Design in Internet Commerce. *American Economic Review, 108* (7), 1820-59.

Dana, J. (1994). Learning in an Equilibrium Search Model. *International Economic Review, 35(3)*: 745–771.

Diamond, P. (1971). A model of price adjustment. *Journal of Economic Theory, 3 (2)*, 156–168.

Fudenberg, D. & Maskin, E. (1986). The Folk Theorem in Repeated Games with Discounting or with Incomplete Information. *Econometrica, 54(3), 533–554*.

Goldfarb, A. & Tucker, C. (2019). Digital economics. *Journal of Economic Literature, 57(1)*, 3–43.

Gordon, R. (2000): A Convergence Theorem for the Riemann Integral. *Mathematics Magazine, 73(2)*, 141–147.

Green, E. J. & Porter, R. H. (1984). Noncooperative Collusion under Imperfect Price Information. *Econometrica, 52(1), 87–100*.

Janssen, M., Moraga-Gonzalez, J. L. & Wildenbeest, M. (2005). Truly Costly Sequential Search and Oligopolistic Pricing. *International Journal of Industrial Organization 23*, 451–466.

Levin, J. D. (2011). The economics of internet markets. *National Bureau of Economic Research*.

Motta, M. (2004). Competition Policy: Theory and Practice, *Cambridge University Press*.

Mukhopadhyay, N. (2000). Probability and Statistical Inference, *New York, Marcel Dekker*.

Montag, F. & Christoph W. (2020). Price transparency against market power. *Available at SSRN 3256476*.

Nilsson, A. (1999). Transparency and Competition. *Stockholm School of Economics Working Paper No. 298*.

Petrikaite, V. (2016). Collusion with Costly Consumer Search. *International Journal of Industrial Organization, 44*, 1-10.

Pennerstorfer, D., Schmidt-Dengler, P., Schutz, N., Weiss, C. & Yontcheva, B. (2020). Information and price dispersion: Theory and evidence. *International Economic Review, 61(2), 871-899*.

Rudin, W. (1976). Principles of mathematical analysis, *New York, McGrawHill*.

Schultz, C. (2017). Collusion in markets with imperfect price information on both sides. *Review of Industrial Organization, 50*, 287–301.

Stahl, D. (1989). Oligopolistic pricing with sequential consumer search. *The American Economic Review*, 700–712.

Tirole, J. (1988). The theory of industrial organization. *MIT press*.

Varian, H. (1980). A model of sales. *The American economic review, 70*, 651–659.